Morse Theory for C*-algebras:

A Geometric Interpretation of Some Noncommutative Manifolds

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Abstract

The approach we present is a modification of the Morse theory for unital C*-algebras. We provide tools for the geometric interpretation of noncommutative CW complexes. These objects were introduced and studied in [2], [7] and [14]. Some examples to illustrate these geometric information in practice are given. A classification of unital C*-algebras by noncommutative CW complexes and the modified Morse functions on them is the main object of this work.

Key words: C-algebra, critical points, CW complexes, homotopy equivalence, homotopy type, Morse function, NCCW complex, poset, pseudo-homotopy type, *-representation, simplicial complex*

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1 Introduction

Among the various approaches in the study of smooth manifolds by the tools from calculus, is the Morse theory. The classical morse theory provides a connection between the topological structure of a manifold M and the topological type of critical points of an open dense family of functions $f : M \to \mathbb{R}$ (the Morse functions).

On a smooth manifold M, a point $a \in M$ is *a critical point* for a smooth function $f : M \to \mathbb{R}$, if the induced map $f_* : T_a(M) \to \mathbb{R}$ is zero. The real number $f(a)$ is then called *a critical value.* The function f is *a Morse function* if i) all the critical values are distinct and ii) its critical points are non degenerate, i.e. the Hessian matrix of second derivatives at the critical points has a non vanishing determinant. The number of negative eigenvalues of this Hessian matrix is *the index of* f at the critical point. The classical Morse theory states as [13]

Theorem : *There exists a Morse function on any differentiable manifold and any differentiable manifold is a CW complex with a λ-cell for each critical point of index λ.*

So once we have information around the critical points of a Morse function on M, we can reconstruct M by a sequence of surgeries.

A C*-algebraic approach which links between operator theory and algebraic geometry, is via a suitable set of equivalence classes of extensions of commutative C*-algebras. This provides a functor from locally compact spaces into abelian groups [7], [11], [14].

If J and B are two C*-algebras, an extension of B by J is a C*-algebra

A together with morphisms $j : J \to A$ and $\eta : A \to B$ such that there is an exact sequence

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\eta} B \tag{1}$$

The aim of the extension problem is the characterization of those C*-algebras A satisfying the above exact sequence. This has something to do with algebraic topology techniques. In the construction of a CW complex, if X_{k-1} is a suitable subcomplex, I^k the unit ball and S^{k-1} its boundary, then the various solutions for the extension problem of $C(X_{k-1})$ by $C_0(I^k - S^{k-1})$ correspond to different ways of attaching I^k to X_{k-1} along S^{k-1}, which means that in the disjoint union $X_{k-1} \cup I^k$ we identify points $x \in S^{k-1}$ with their image $\varphi_k(x)$ under some attaching map $\varphi_k : S^{k-1} \to X_{k-1}$.

After the construction of noncommutative geometry [1], there have been attempts for the formulation of classical tools of differential geometry and topology in terms of C*-algebras (in some sense the dualization of the notions, [3], [4], [10]. The dual concept of CW complexes , with some regards, is the notion of noncommutative CW complexes [7] and [14]. Our approach in this work is the geometric study of these structures. So many works are done on the combinatorial structures of noncommutative simplicial complexes and their decompositions, for example [2], [6], [8], [9]. Following these works, together with some topological constructions , we show how a modification of the classical Morse theory to the level of C*-algebras will provide an innovative way to explain the geometry of noncommutative CW complexes through the critical ideals of the modified Morse function. This leads to some classification theory.

This paper is prepared as follows. After an introduction to the notion

of primitive spectrum of a C*-algebra, it will proceed the topological structure in detail and present some instantiation. In the continue we study the noncommutative CW complexes and interpret their geometry by introducing the modified Morse function. All these provide tools for the modified Morse theory for C*-algebras. The last section devoted to the prove of this theorem. It states as

Main Theorem : *Every unital C*-algebra with an acceptable Morse function on it is of pseudo-homotopy type of a noncommutative CW complex, having a k-th decomposition cell for each critical chain of order k.*

2 The Structure of the Primitive Spectrum

The technique we follow to link between the geometry, topology and algebra is the primitive spectrum point of view. In fact as we will see in our case it is a promissive candidate for the noncommutative analogue of a topological manifold M. We review some preliminaries on the primitive spectrum. Details can be found in [5], [10], [12].

Let M be a compact topological manifold and $A = C(M)$ be the commutative unital C*-algebra of continuous functions on M. The *primitive spectrum* of A is the space of kernels of irreducible *-representations of A. It is denoted by $Prim(A)$. The topology on this space is given by the closure operation as follows:

For any subset $X \subseteq Prim(A)$, the closure of X is defined by

$$\bar{X} := \{I \in Prim(A) : \bigcap_{J \in X} J \subset I\} \qquad (2)$$

This operation defines a topology on $Prim(A)$ (the hull-kernel topology), making it into a T_0-space.

Definition 2.1. A subset $X \subset Prim(A)$ is called *absorbing* if it satisfies the following condition

$$I \in X, I \subseteq J \Rightarrow J \in X \tag{3}$$

Lemma 2.2. *The closed subsets of $Prim(A)$ are exactly its absorbing subsets.*

Proof. It is clear from the very definition of closed sets. ☐

For each $x \in M$ let

$$I_x := \{f \in A : f(x) = 0\}$$

I_x is a closed maximal ideal of A. It is in fact the kernel of the evaluation map

$$(ev)_x : A \to \mathbb{C}$$

$$f \rightsquigarrow f(x)$$

This provides a homeomorphism

$$I : M \to Prim(A) \tag{4}$$

between M and $Prim(A)$, defined by $I(x) := I_x$.

To each $I \in Prim(A)$, there corresponds an absorbing set

$$W_I := \{J \in Prim(A) : J \supseteq I\}$$

and an open set

$$O_I := \{J \in Prim(A) : J \subseteq I\}$$

containing I.

Being a T_0-space, $Prim(A)$ can be made into a partially ordered set (poset) by setting for $I, J \in Prim(A)$,

$$I < J \Leftrightarrow I \subset J$$

which is equivalent to $O_I \subset O_J$ or $W_I \supset W_J$.

The topology of $Prim(A)$ can be given equivalently by means of this partial order

$$I < J \Leftrightarrow J \in \{\bar{I}\}$$

where $\{\bar{I}\}$ is the closure of the one point set $\{I\}$.

Now let A be an arbitrary unital C*-algebra. Since A is unital, then $Prim(A)$ is compact. Let $Prim(A) = \bigcup_{i=1}^{n} O_{I_i}$ be a finite open covering.

An equivalence relation on $Prim(A)$ is given by

$$I \sim J \Leftrightarrow J \in O_I (I \in O_J)$$

In each O_{I_i} choose one I_i with respect to the above equivalence relation.

Let $I_1, I_2, ...I_m$ be chosen this way so that $Prim(A)$ is made into a finite lattice for which the points are the equivalence classes of $[I_1], ...[I_m]$. For simplicity we show each class $[I_i]$ by its representative I_i. Let

$$J_{i_1,....i_k} := I_{i_1} \cap ... \cap I_{i_k}$$

where $1 \leq i_1, ...i_k \leq m, 1 \leq k \leq m$

Set

$$W_{i_1,....i_k} := \{J \in Prim(A) : J \supset J_{i_1,....i_k}\}$$

This is a closed subset of $Prim(A)$.

Definition 2.3. When $J_{i_1,\dots.i_k} \neq 0$, then it is called a *k-ideal* in A and its corresponding closed set $W_{i_1,\dots.i_k}$ in $Prim(A)$ is called a $k-chain$.

Remark 2.4. If for some $1 \leq i_1, ...i_k \leq m, 1 \leq k \leq m$, we have $J_{i_1,\dots.i_k} = 0$, then $W_{i_1,\dots.i_k} = Prim(A)$. Also for each pair of indices $(i_1, \dots.i_k), \sigma(i_1, \dots.i_{k+1})$

$$W_{i_1,\dots.i_k} \subseteq W_{\sigma(i_1,\dots.i_{k+1})}$$

where σ is a permutation on k+1 elements.

Remark 2.5. Let

$$X_0 \subset X_1 \subset ... \subset X_n = X$$

be an n-dimensional CW complex structure for a topological space X, so that X_0 is a finite discrete space consisting of 0-cells, and for $k = 1, ..., n$ each k-skeleton X_k is obtained by attaching λ_k number of k-disks to X_{k-1} via the attaching maps

$$\varphi_k : \bigcup_{\lambda_k} S^{k-1} \to X_{k-1}$$

In other words

$$X_k = \frac{X_{k-1} \bigcup (\cup_{\lambda_k} I^k)}{x \sim \varphi_k(x)} := X_{k-1} \bigcup_{\varphi_k} (\cup_{\lambda_k} I^k) \tag{5}$$

wherever $x \in S^{k-1}$, where $I^k := [0,1]^k$ and $S^{k-1} := \partial I^k$. The quotient map is denoted by

$$\rho : X_{k-1} \bigcup (\cup_{\lambda_k} I^k) \to X_k$$

For more details see [11].

A cell complex structure is induced on $Prim(C(X))$ by the following procedure:

Let $A_k = C(X_k),\ k = 1, ..., n$. For each 0-cell C_0 in X_0, let I_{C_0} be its image under the homeomorphism $I : X_0 \to Prim(C(X_0))$ of relation (4). By considering the restriction of functions on X to X_0, I_{C_0}s will be the 0-ideals for $A = C(X)$ and

$$W_{C_0} := \{J \in Prim(A) : I_{C_0} \subset J\}$$

the 0-chains for $Prim(A)$.

The 1-ideals are of the form $I_{C_1} := \bigcap_{x \in C_1} I_x$ with the corresponding 1-chains

$$W_{C_1} := \{J \in Prim(A) : I_{C_1} \subset J\}$$

In the same way the k-ideals are $I_{C_k} = \bigcap_{x \in C_k} I_x$ for $2 \leq k \leq n$, with the corresponding k-chains

$$W_{C_k} := \{J \in Prim(A) : I_{C_k} \subset J\}$$

(An ideal in A_{k-1} can be thought of as an ideal in A_k by the restriction of functions.)

In the following two examples we identify the k-ideals and the k-chains for the CW complex structures of the closed interval [0,1] and the 2-torus $S^1 \times S^1$.

Example 2.6. Let $X_0 = \{0, 1\}$ and $X_1 = [0, 1]$ be the zero and one skeleton for a CW complex structure of [0,1]. $A_0 = C(X_0) \simeq \mathbb{C} \oplus \mathbb{C}$ and $A_1 = C(X_1)$ and the 0-ideals I_0 and I_1 and their corresponding 0-chains W_0 and W_1 are

$$I_0 = \{f \in A_0 : f(0) = 0\} \simeq \mathbb{C}, I_1 = \{f \in A_0 : f(1) = 0\} \simeq \mathbb{C}$$

and

$$W_0 = \{J \in PrimA_0 : I_0 \subset J\} \simeq \{0\}, W_1 = \{J \in PrimA_0 : I_1 \subset J\} \simeq \{1\}$$

For the only 1-ideal we have

$$I = I_0 \cap I_1 = 0$$

with the corresponding 1-chain

$$W_I = \{J \in Prim(A) : I \subset J\} = Prim(A) \simeq [0, 1]$$

Example 2.7. Let

$$X_0 = \{0\}, X_1 = \{\alpha, \beta\}, X_2 = T^2 = S^1 \times S^1$$

be the skeletons for a CW complex structure for the 2-torus T^2. α, β are homeomorphic images of S^1 (closed curves with the origin 0). Let $A_0 = C(X_0) = \mathbb{C}$, $A_1 = C(X_1)$ and $A_2 = A = C(T^2)$.

The 0-ideal and its corresponding 0-chain are

$$I_0 = \{f \in A_0 : f(0) = 0\}$$

$$W_0 = \{J \in Prim(A_0) : I_0 \subset J\} \simeq Prim(A_0) = \{0\}$$

Also the 1-ideals I_1, I_2 and 1-chains W_{I_1}, W_{I_2} are

$$I_1 = \{f \in A_1 : f(\alpha) = 0\} = \cap_{x \in \alpha} I_x$$

$$I_2 = \{f \in A_1 : f(\beta) = 0\} = \cap_{x \in \beta} I_x$$

$$W_{I_1} = \{J \in Prim(A_1) : I_1 \subset J\} \simeq \alpha$$

$$W_{I_2} = \{J \in Prim(A_1) : I_2 \subset J\} \simeq \beta$$

Finally the only 2-ideal and its corresponding 2-chain are

$$I = \{f \in A : f(T^2) = 0\} \simeq 0$$

$$W_I = \{J \in Prim(A) : I \subset J\} \simeq T^2$$

3 The Noncommutative CW Complexes (NCCW Complexes)

In this section we see how the construction of the primitive spectrum of the previous section helps us to study the noncommutative CW complexes.

For a continuous map $\phi : X \to Y$ between topological spaces X and Y, the C*-morphism induced on their associated C*-algebras is denoted by $C(\phi) : C(Y) \to C(X)$ and is defined by $C(\phi)(g) := go\phi$ for $g \in C(Y)$.

Definition 3.1. Let A_1, A_2 and C be C*-algebras. *A pull back for C via morphisms $\alpha_1 : A_1 \to C$ and $\alpha_2 : A_2 \to C$* is the C*-subalgebra of $A_1 \oplus A_2$ denoted by $PB(C, \alpha_1, \alpha_2)$ defined by

$$PB(C, \alpha_1, \alpha_2) := \{a_1 \oplus a_2 \in A_1 \oplus A_2 : \alpha_1(a_1) = \alpha_2(a_2)\}$$

For any C*-algebra A, let

$$S^n A := C(S^n \to A), I^n A := C([0,1]^n \to A), I_0^n A := C_0((0,1)^n \to A)$$

where S^n is the n-dimensional unit sphere.

We review the definition of noncommutative CW complexes from [7], [14].

Definition 3.2. A *0-dimensional noncommutative CW complex* is any finite dimensional C*-algebra A_0. Recursively an *n-dimensional noncommutative CW complex (NCCW complex)* is any C*-algebra appearing in the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_0^n F_n & \longrightarrow & A_n & \xrightarrow{\pi} & A_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \varphi_n} & & \\
0 & \longrightarrow & I_0^n F_n & \longrightarrow & I^n F_n & \xrightarrow{\delta} & S^{n-1} F_n & \longrightarrow & 0
\end{array} \tag{6}$$

Where the rows are extensions, A_{n-1} an (n-1)-dimensional noncommutative CW complex, F_n some finite dimensional C*-algebra of dimension λ_n, δ the boundary restriction map, φ_n an arbitrary morphism (called the connecting morphism), for which

$$A_n = PB(S^{n-1}F_n, \delta, \varphi_n) := \{(\alpha, \beta) \in I^n F_n \oplus A_{n-1} : \delta(\alpha) = \varphi_n(\beta)\} \tag{7}$$

f_n and π are respectively projections onto the first and second coordinates.

With these notations $\{A_0, ..., A_n\}$ is called *the noncommutative CW complex decomposition of dimension n for* $A = A_n$

For each $k = 0, 1, ...n$, A_k is called *the k-th decomposition cell.*

Proposition 3.3. *Let X be an n-dimensional CW complex containing cells of each dimension $\leq n$. Then there exists a noncommutative CW complex decomposition of dimension n for $A = C(X)$.*

Conversely suppose $\{A_0, ..., A_n\}$ be a noncommutative CW complex decomposition of dimension n for the C-algebra A such that A and all the $A_i s$ $(i = 0, .., n)$ are unital. For each $k \leq n$, let $X_k = Prim(A_k)$. Then there exists an n-dimensional CW complex structure on $Prim(A)$ with X_k as its k-skeleton for each $k \leq n$.*

Proof. Let

$$X_0 \subset X_1 \subset ... \subset X_n = X$$

be a CW complex structure for X where for each $k \leq n$, X_k is the k-skeleton defined in relation (5). For each $k = 0, ..., n$, let $A_k = C(X_k)$,

$i : \bigcup_{\lambda_k} S^{k-1} \to \bigcup_{\lambda_k} I^k$ be the injection, and $\varphi_k : \bigcup_{\lambda_k} S^{k-1} \to X_{k-1}$ be the attaching maps. Furthermore let $C(i)$ and $C(\varphi)$ be their induced maps. Let

$$PB := PB(C(\bigcup_{\lambda_k} S^{k-1}), C(\varphi_k), C(i))$$

Define

$$\theta : C(X_k) \to PB$$

by $\theta(f) = (fo\rho)_1 \oplus (fo\rho)_2$ for $f \in C(X_k)$.

Where $(fo\rho)_1$ is the restriction of $(fo\rho)$ to $\bigcup_{\lambda_k} I^k$ and $(fo\rho)_2$ is the restriction of $(fo\rho)$ to X_{k-1}.

θ is well defined since $C(\varphi_k)((fo\rho)_1) = C(i)((fo\rho)_2)$. Also for $(h, g) \in PB$, we have $C(\varphi_k)(h) = C(i)(g)$ and so if $f \in C(X_k)$ be defined by $f(y) = g(y)$ for $y \in \bigcup_{\lambda_k} I^k$ and $f(y) = h(y)$ for $y \in X_{k-1}$, then $\theta(f) = (h, g)$.

Now the noncommutative CW complex decomposition of dimension n for $A = C(X)$ is given by $\{A_0, ..., A_n\}$.

Conversely let A_n be as in (7). Let

$$\varphi_n^* : S^{n-1} \to Prim(A_{n-1})$$

be the attaching map induced by the connecting morphism

$$\varphi_n : A_{n-1} \to S^{n-1}F_n$$

of diagram (6). Then using the notation in relation (5),

$$Prim(A_n) = Prim(A_{n-1}) \cup_{\varphi_n^*} I^n$$

We note that $\varphi_n^* = C(\varphi_n)$. Furthermore for $k \leq n$, $\varphi_k^*(S^{k-1})$ is a closed subset of $Prim(A_{k-1})$. It is of the form

$$\varphi_k^*(S^{k-1}) = \{J \in Prim(A_{k-1}) : I_{k-1} \subset J\}$$

for some ideal I_{k-1} in A_{k-1}. In fact

$$I_{k-1} = \bigcap_{J \in \varphi_k^*(S^{k-1})} J$$

$\square$

Example 3.4. Following the notations of diagram (6), a 1-dimensional noncommutative CW complex decomposition for $A = C([0,1]) = C(I)$ is given by

$$A_0 = \mathbb{C} \oplus \mathbb{C}, A_1 = C([0,1])$$

Let $F_1 = \mathbb{C}$, then

$$I_0^1 F_1 = C_0((0,1)), I^1 F_1 = C([0,1]), S^0 F_1 = \mathbb{C} \oplus \mathbb{C}$$

$\varphi_1 = id$. Also

$$C(I) = PB(S^0 F_1, \delta, \varphi_1) = \{f \oplus (\lambda \oplus \mu) \in C([0,1]) \oplus (\mathbb{C} \oplus \mathbb{C}) : f(0) = \lambda, f(1) = \mu\}$$

together with the maps

$\pi : A_1 \to A_0$ defined by $\pi(f \oplus (\lambda \oplus \mu)) = \lambda \oplus \mu$ and $f_1 : A_1 \to I^1 F_1 = A_1$ defined by $f_1(f \oplus (\lambda \oplus \mu)) = f$ and finally $\delta : I^1 F_1 = A_1 \to S^0 F_1 = \mathbb{C} \oplus \mathbb{C}$ defined by $\delta(f) = f(0) \oplus f(1)$.

4 Modified Morse Theory on C*-Algebras

In this section, following the study of the Morse theory for the cell complexes in [2], [6], [8], [9], with some modification, we define the Morse function for the C*-algebras and state and prove the modified Morse theory for the noncommutative CW complexes. This is a classification theory in the category of C*-algebras and noncommutative CW complexes.

Definition 4.1. Let A and B be C*-algebras. Two morphisms $\alpha, \beta : A \to B$ are said to be *homotopic* if there exists a family $\{H_t\}_{t\in[0,1]}$ of morphisms $H_t : A \to B$ such that for each $a \in A$ the map $t \mapsto H_t(a)$ is a norm continuous path in B with $H_0 = \alpha$ and $H_1 = \beta$. In this case we write $\alpha \sim \beta$.

C*-algebras A and B are called *of the same homotopy type* if there are morphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi o \psi \sim id_B$ and $\psi o \varphi \sim id_A$. In this case the morphisms φ and ψ are called *homotopy equivalent.*

Definition 4.2. Let A and B be unital C*-algebras. We say A is *of pseudo-homotopy type* of B if $C(Prim(A))$ and B are of the same homotopy type.

Remark 4.3. In the case of unital commutative C*-algebras, by the GNS construction, $C(Prim(A)) = A$, [10]. So the notions of pseudo-homotopy type and the same homotopy type are equivalent.

Let A be a unital C*-algebra and

$$\Sigma = \{\{W_{i_1,\dots,i_k}\}_{1\leq k\leq n}\}_{1\leq i_1,\dots,i_k\leq n}$$

be the set of all k-chains (k=1,...,n) in $Prim(A)$.

Lemma 4.4. *Let*

$$\Gamma = \{\{I_{i_1,\dots,i_k}\}_{1\leq k\leq n}\}_{1\leq i_1,\dots,i_k\leq n}$$

be the set of all k-ideals corresponding to the k-chains of Σ *for k=1,...,n, then* Γ *is an absorbing set.*

Proof. This follows from the fact that for each $I_{i_1,\dots,i_k} \in \Gamma$ and for each $J \in \Gamma$, the relation $I_{i_1,\dots,i_k} \subset J$ is equivalent to $J = I_{i_1,\dots,i_t}$ for some $t \leq k$ meaning $J \in \Gamma$. □

Definition 4.5. Let $f : \Sigma \to \mathbb{R}$ be a function. The k-chain $W_k = W_{i_1,\ldots,i_k}$ is called *a critical chain of order k* for f, if for each (k+1)-chain W_{k+1} containing W_k and for each (k-1)-chain W_{k-1} contained in W_k, we have

$$f(W_{k+1}) \geq f(W_k), f(W_{k-1}) \leq f(W_k)$$

The corresponding ideal I_k to W_k is called *the critical ideal of order k.*

Definition 4.6. A function $f : \Sigma \to \mathbb{R}$ is called *a modified Morse function* on the C*-algebra A, if for each k-chain W_k in Σ, there is at most one (k+1)-chain W_{k+1} containing W_k and at most one (k-1)-chain W_{k-1} contained in W_k, such that

$$f(W_{k+1}) \leq f(W_k), f(W_{k-1}) \geq f(W_k)$$

f is called *acceptable Morse function* if for each k, if f has a critical chain of order k, then there exists critical chains of order i for all $i \leq k$.

Now we state our main theorem. This geometric condition for a C*-algebra to admit a noncommutative CW complex decomposition classifies specific unital C*-algebras up to pseudo-homotopy type.

Theorem 4.7. *Every unital C*-algebra A with an acceptable modified Morse function f on it, is of pseudo-homotopy type of a noncommutative CW complex having a k-th decomposition cell for each critical chain of order k.*

Before starting the proof of this theorem, we state the discrete Morse theory of Forman from [8] and state our modification of it.

Theorem (Discrete Morse Theory): Suppose Δ is a simplicial complex with a discrete Morse function. Then Δ is homotopy equivalent to a CW complex with one cell of dimension p for each critical p-simplex [8].

Lemma 4.8. *If f is an acceptable modified Morse function on A, then $Prim(A)$ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical chain of order p.*

Proof. In the discrete Morse theory it suffices to substitute Γ for the simplicial complex Δ. Since Γ is absorbing, it satisfies the properties of the simplicial complex Δ in the discrete Morse theorem. It follows that $Prim(A)$ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical chain of order p. □

Now we start the proof of the main theorem.

Proof. When A is a unital C*-algebra, then the acceptable modified Morse function on A is in fact a function on the simplicial complex of all k-ideals in $Prim(A)$ (a function on Γ). From lemma 4.8 we conclude that $Prim(A)$ is of homotopy type of a finite CW complex Ω. From the proposition 3.3 there is a noncommutative complex decomposition for $C(\Omega)$ making it into a noncommutative CW complex. Now $C(Prim(A))$and $C(\Omega)$ are C*-algebras of the same homotopy type, which means A is of psuedo-homotopy type of the noncommutative CW complex $C(\Omega)$. Furthermore since f is acceptable, from the proof of proposition 3.3 it follows that when there exists a critical k-chain for f, then there exists C*-algebras A_i for each $i \leq k$ so that $\{A_0, ..., A_k\}$ is a noncommutative CW complex decomposition for $C(Prim(A))$ yielding a noncommutative CW complex decomposition for $C(\Omega)$.

□

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